Bayesian Matrix Completion via Adaptive Relaxed Spectral Regularization

Yang Song†, Jun Zhu‡

†Department of Physics, Tsinghua University
‡Department of Computer Science & Tech., Tsinghua University

Feb. 14, 2016

Low-rank Matrix Completion

List of symbols

Z_{mn}:	Latent data matrix
$\Omega \subset [m] \times [n]$:	Indices of observed entries
$P_\Omega(X)$:	Observed (noisy) entries

With the low-rank assumption $\mathrm{rank}(Z) \ll \min(m, n)$,

$$\min_Z \mathcal{L}(P_\Omega(X), Z) + \lambda \, \mathrm{rank}(Z). \quad (1)$$

A ubiquitous convex surrogate:

$$\min_Z \mathcal{L}(P_\Omega(X), Z) + \lambda \left\| Z \right\|_* . \quad (2)$$

Probabilistic Matrix Completion via Matrix Factorization

Adopt Gaussian observation noise,

$$\max_Z \frac{1}{C_1} \exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\} \cdot \frac{1}{C_2} \exp\{\lambda \|Z\|_*\}. \quad (3)$$

Probabilistic Matrix Completion via Matrix Factorization

Adopt Gaussian observation noise,

$$\max_Z \frac{1}{C_1} \exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\} \cdot \frac{1}{C_2} \exp\{\lambda \|Z\|_*\}. \quad (3)$$

$$\boxed{\|Z\|_* = \min_{Z = U^\intercal V} \frac{1}{2}(\|U\|_F^2 + \|V\|_F^2)}. \quad (4)$$

which inspires probabilistic matrix factorization with Gaussian priors:

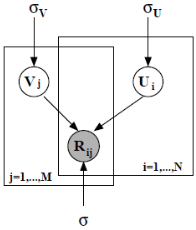
Probabilistic Matrix Completion via Matrix Factorization

Adopt Gaussian observation noise,

$$\max_{Z} \frac{1}{C_1} \exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\} \cdot \frac{1}{C_2} \exp\{\lambda \|Z\|_*\}. \qquad (3)$$

$$\boxed{\|Z\|_* = \min_{Z=U^\intercal V} \frac{1}{2}(\|U\|_F^2 + \|V\|_F^2)}. \qquad (4)$$

which inspires probabilistic matrix factorization with Gaussian priors:



Probabilistic Matrix Completion Directly via Nuclear Norm?

$$\max_{Z} \frac{1}{C_1} \underbrace{\exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\}}_{\text{Gaussian distributions}} \cdot \frac{1}{C_2} \underbrace{\exp\{\lambda \|Z\|_*\}}_{?}$$

$$= \max_{Z} \frac{1}{C_1} \exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\} \cdot \frac{1}{C_2} \underbrace{\exp\{\sum_{k=1}^{r} d_k\}}_{\text{Exponential distributions}}$$

$$= \max_{\mathbf{u}_{1:k}, \mathbf{v}_{1:k}, d_{1:k}} p(\mathbf{u}_{1:k}, \mathbf{v}_{1:k}, d_{1:k} | P_\Omega(X))$$

where $Z = \sum_{k=1}^{r} d_k \mathbf{u}_k \mathbf{v}_k^\intercal$ is the singular value decomposition (SVD).

Probabilistic Matrix Completion Directly via Nuclear Norm?

$$\max_{Z} \frac{1}{C_1} \underbrace{\exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\}}_{\text{Gaussian distributions}} \cdot \frac{1}{C_2} \underbrace{\exp\{\lambda \|Z\|_*\}}_{?}$$

$$= \max_{Z} \frac{1}{C_1} \exp\{\frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2\} \cdot \frac{1}{C_2} \underbrace{\exp\{\sum_{k=1}^{r} d_k\}}_{\text{Exponential distributions}}$$

$$= \max_{\mathbf{u}_{1:k}, \mathbf{v}_{1:k}, d_{1:k}} p(\mathbf{u}_{1:k}, \mathbf{v}_{1:k}, d_{1:k} | P_\Omega(X))$$

where $Z = \sum_{k=1}^{r} d_k \mathbf{u}_k \mathbf{v}_k^\intercal$ is the singular value decomposition (SVD).

$$\boxed{\begin{aligned} \mathbf{u}_i^\intercal \mathbf{u}_j &= \delta_{ij} \\ \mathbf{v}_i^\intercal \mathbf{v}_j &= \delta_{ij} \end{aligned}} \qquad (5)$$

Probabilistic Matrix Completion Directly via Nuclear Norm?

The factorized prior $p(Z) = p(\vec{d})p(U)p(V)$ needs to deal with *Stiefel Manifolds*.

Probabilistic Matrix Completion Directly via Nuclear Norm?

The factorized prior $p(Z) = p(\vec{d})p(U)p(V)$ needs to deal with *Stiefel Manifolds*.

$$\min_{\vec{d},U,V} \frac{1}{2\sigma^2} \left\| P_\Omega \left(X - \sum_{k=1}^{r} d_k \mathbf{u}_k \mathbf{v}_k^\intercal \right) \right\|_F^2 + \lambda \sum_{k=1}^{r} d_k \tag{6}$$

$$s.t. \quad d_k \geq 0, \quad \mathbf{u}_i^\intercal \mathbf{u}_j = \delta_{ij}, \quad \mathbf{v}_i^\intercal \mathbf{v}_j = \delta_{ij} \tag{7}$$

Probabilistic Matrix Completion Directly via Nuclear Norm?

The factorized prior $p(Z) = p(\vec{d})p(U)p(V)$ needs to deal with *Stiefel Manifolds*.

$$\min_{\vec{d},U,V} \frac{1}{2\sigma^2} \left\| P_\Omega \left(X - \sum_{k=1}^{r} d_k \mathbf{u}_k \mathbf{v}_k^\intercal \right) \right\|_F^2 + \lambda \sum_{k=1}^{r} d_k \quad (6)$$

$$s.t. \quad d_k \geq 0, \quad \mathbf{u}_i^\intercal \mathbf{u}_j = \delta_{ij}, \quad \mathbf{v}_i^\intercal \mathbf{v}_j = \delta_{ij} \quad (7)$$

Is there anything like matrix factorization to solve the problem more easily?

What are the benefits?

What are the benefits of using spectral regularization compared to matrix factorization?

- Direct control over the singular values. Incorporate abundant information via priors.
- Intuitive rank inference.

Relaxed Spectral Regularization

Theorem

The unit ball constraints $\|\mathbf{u}_k\| \leq 1$ and $\|\mathbf{v}_k\| \leq 1$ are sufficient to get the same MAP value and solutions.

$$\min_{Z} \frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2 + \lambda \|Z\|_* \tag{8}$$

$$\Leftrightarrow \min_{\vec{d},A,B} \frac{1}{2\sigma^2} \left\| P_\Omega \left(X - \sum_{k=1}^{r} d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal \right) \right\|_F^2 + \lambda \sum_{k=1}^{r} d_k \tag{9}$$

$$s.t. \quad d_k \geq 0, \quad \|\boldsymbol{\alpha}_k\| \leq 1, \quad \|\boldsymbol{\beta}_k\| \leq 1 \tag{10}$$

Relaxed Spectral Regularization

Theorem

The unit ball constraints $\|\mathbf{u}_k\| \leq 1$ and $\|\mathbf{v}_k\| \leq 1$ are sufficient to get the same MAP value and solutions.

$$\min_{Z} \frac{1}{2\sigma^2} \|P_\Omega(Z - X)\|_F^2 + \lambda \|Z\|_* \tag{8}$$

$$\Leftrightarrow \min_{\vec{d},A,B} \frac{1}{2\sigma^2} \left\| P_\Omega \left(X - \sum_{k=1}^{r} d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal \right) \right\|_F^2 + \lambda \sum_{k=1}^{r} d_k \tag{9}$$

$$s.t. \quad d_k \geq 0, \quad \|\boldsymbol{\alpha}_k\| \leq 1, \quad \|\boldsymbol{\beta}_k\| \leq 1 \tag{10}$$

Note

Relaxed Spectral Regularization $\Leftrightarrow$ Spectral Regularization.

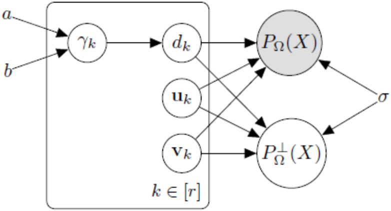
Adaptive Relaxed Spectral Regularization

A more adaptive version. Salakhutdinov et. al did similar modification for probabilistic matrix factorization.

$$\min_{\vec{d},U,V} \underbrace{\frac{1}{2\sigma^2}\left\| P_\Omega\left(X - \sum_{k=1}^{r} d_k \mathbf{u}_k \mathbf{v}_k^\intercal\right)\right\|_F^2}_{\text{Truncated Gaussian distributions}} + \underbrace{\sum_{k=1}^{r} \gamma_k d_k}_{\text{Exponential distributions}} \tag{11}$$

$$s.t. \quad d_k \geq 0, \quad \|\mathbf{u}_k\| \leq 1, \quad \|\mathbf{v}_k\| \leq 1. \tag{12}$$

Graphical Model



$$\tilde{p}(\mathbf{u}_k) = \begin{cases} 1, & \|\mathbf{u}_k\| \le 1 \\ 0, & \|\mathbf{u}_k\| > 1 \end{cases} \tag{13}$$

$$\tilde{p}(\mathbf{v}_k) = \begin{cases} 1, & \|\mathbf{v}_k\| \le 1 \\ 0, & \|\mathbf{v}_k\| > 1 \end{cases} \tag{14}$$

$$p(d_k|\gamma_k) = \gamma_k e^{-\gamma_k d_k} \tag{15}$$

$$p(\gamma_k) \propto \gamma_k^{a-1} e^{-b\gamma_k} \tag{16}$$

Inference

1. Repeat
   - Sample $\vec{\gamma}$,$\vec{d}$,U and V consecutively with Gibbs sampling.
   - Update a,b and σ with Monte Carlo EM using the posterior samples got in the previous step.
2. Prediction:

$$x_{ij} = \langle \sum_{k=1}^{r} d_k u_{ki} v_{kj} \rangle \tag{17}$$

Rank Recovery

Counting the number of non-zero d_k's, but with slight modification.

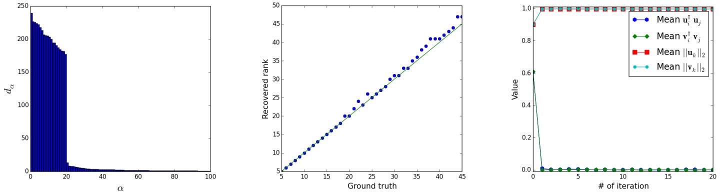


Figure: (a) The recovered rank of a 200×200 synthetic matrix with rank 20. (b) Rank recovery results on synthetic data, with solid line representing the ground truth. (c) Vectors tend to get orthonormalized.

MovieLens and EachMovie

Table: Experimental results of various methods on the MovieLens and EachMovie datasets.

	MovieLens		EachMovie	
Algorithm	**NMAE**	**RMSE**	**NMAE**	**RMSE**
M^3F	0.1652 ± 0.0004	0.9644 ± 0.0012	0.1932 ± 0.0003	1.4598 ± 0.0019
iPM^3F	0.1604 ± 0.0004	0.9386 ± 0.0024	0.1819 ± 0.0006	1.3760 ± 0.0034
SoftImpute	0.1829 ± 0.0002	0.9469 ± 0.0009	0.2039 ± 0.0002	1.2948 ± 0.0037
SoftImpute-ALS	0.1783 ± 0.0001	0.9196 ± 0.0013	0.2018 ± 0.0004	1.2757 ± 0.0008
HASI	0.1813 ± 0.0002	0.9444 ± 0.0011	0.1992 ± 0.0003	1.2612 ± 0.0016
BPMF	0.1663 ± 0.0002	0.8460 ± 0.0006	0.2012 ± 0.0001	1.2363 ± 0.0007
GASR	0.1673 ± 0.0005	0.8528 ± 0.0025	0.1930 ± 0.0009	1.2015 ± 0.0044

Different Missing Rates

Table: Results on different missing rates

Setting	$m = 500, n = 500, r = 30, q = 5$			
Missing-Rates	90%	80%	50%	0%
BPMF	1.6842 ± 0.1374	0.3210 ± 0.0168	0.1304 ± 0.0022	0.0933 ± 0.0000
GASR	0.1992 ± 0.0241	0.1321 ± 0.0086	0.0841 ± 0.0028	0.0724 ± 0.0036
Setting	$m = 1000, n = 1000, r = 50, q = 10$			
Missing-Rates	90%	80%	50%	0%
BPMF	0.9422 ± 0.0478	0.2396 ± 0.0033	0.1105 ± 0.0013	0.0859 ± 0.0007
GASR	0.2513 ± 0.0045	0.1688 ± 0.0041	0.1270 ± 0.0034	0.1115 ± 0.0057

Thank You!

Thank you for your time! Any questions?