Bayesian Matrix Completion via Adaptive Relaxed Spectral Regularization: Supplementary Materials

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Proof for Theorem 1

Proof. Denote $Z = \sum_{k=1}^r d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal$ and conduct singular value decomposition to give

$$Z = \sum_{k=1}^r \sigma_k \mathbf{u}_k \mathbf{v}_k^\intercal = \sum_{k=1}^r d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal \tag{1}$$

where $\sigma_{1:r}$ are singular values of Z and $U = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m\}$ and $V = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ are corresponding orthogonal matrices. Note that we denote $r = \min(m,n)$ throughout this paper. If the true rank is smaller than $\min(m,n)$, then singular values with indices larger than the rank are assumed to be zero. We will try to prove

$$\sum_{k=1}^r \sigma_k \leq \sum_{k=1}^r d_k, \tag{2}$$

which actually implies all the assertions in the theorem.

For $\forall i \in [r]$, left multiply equation (1) with $\mathbf{u}_i^\intercal$ and right multiply with $\mathbf{v}_i$ to obtain

$$\sigma_i = \sum_{k=1}^r d_k \mathbf{u}_i^\intercal \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal \mathbf{v}_i. \tag{3}$$

Since U and V are orthogonal matrices with full ranks in a singular value decomposition, we can regard column vectors of U and V to be eigenbases of space $\mathbb{R}^m$ and $\mathbb{R}^n$. Hence it is natural to obtain

$$\boldsymbol{\alpha}_i = \sum_{j=1}^m x_{ij} \mathbf{u}_j, \quad \boldsymbol{\beta}_i = \sum_{j=1}^n y_{ij} \mathbf{v}_j, \tag{4}$$

where $\forall i \in [r], \quad \sum_{j=1}^m x_{ij}^2 \leq 1$ and $\sum_{j=1}^n y_{ij}^2 \leq 1$.

We then rewrite Eq. (3) to give

$$\sigma_i = \sum_{k=1}^r d_k x_{ki} y_{ki}. \tag{5}$$

As a result,

$$\begin{aligned}
\sum_{i=1}^r \sigma_i &= \sum_{i=1}^r \sum_{k=1}^r d_k x_{ki} y_{ki} = \sum_{k=1}^r d_k \sum_{i=1}^r x_{ki} y_{ki} \\
&\leq \sum_{k=1}^r d_k \left(\sum_{i=1}^r x_{ki}^2 \right)^{1/2} \left(\sum_{i=1}^r y_{ki}^2 \right)^{1/2} \\
&\leq \sum_{k=1}^r d_k \left(\sum_{i=1}^m x_{ki}^2 \right)^{1/2} \left(\sum_{i=1}^n y_{ki}^2 \right)^{1/2} \\
&\leq \sum_{k=1}^r d_k,
\end{aligned} \tag{6}$$

which means for any valid tuples of $(d_{1:r}, \boldsymbol{\alpha}_{1:r}, \boldsymbol{\beta}_{1:r})$, replacing $\sum_{k=1}^r d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal$ with $\sum_{k=1}^r \sigma_k \mathbf{u}_k \mathbf{v}_k^\intercal$ in P2, according to (2), will not make the solution worse. This indicates that there is at least one optimal solution of P2 having the form of singular value decompositions like those in P1$'$.

As a result, we always have $s \leq t$, because for any optimal solution of P2, we can get an SVD form compatible to the constraints of P1$'$ with an objective value not larger. However, considering the fact that P2 is basically the same problem as P1$'$ with looser constraints, we conclude that $t \leq s$. Following the reasoning above we get $s \leq t$ and $s \geq t$, which exactly means $s = t$.

Suppose we have got the optimal solutions of P2, which is denoted as $(d_{1:r}^*, \boldsymbol{\alpha}_{1:r}^*, \boldsymbol{\beta}_{1:r}^*)$. We assert that $Z^* = \sum_{k=1}^r d_k^* \boldsymbol{\alpha}_k^* \boldsymbol{\beta}_k^{*\intercal}$ is the optimal solution of P1, because plugging Z^* into P1 will yield a value not greater than t. Since $s = t$ and s is the minimum possible value of P1, we conclude that plugging Z^* into P1 gets the value s, which means Z^* is the optimal solution for P1.

Similarly, suppose that the optimal solution of P1 is $Z^\dagger$, we compute its singular value decomposition to get $Z^\dagger = \sum_{k=1}^r \sigma_k^\dagger \mathbf{u}_k^\dagger \mathbf{v}_k^{\dagger\intercal}$. Then plugging $(\sigma_{1:r}^\dagger, \mathbf{u}_{1:r}^\dagger, \mathbf{v}_{1:r}^\dagger)$ into P2 will give the value s. Since $s = t$, we conclude that $Z^\dagger$ is an optimal solution for P2.

Note that it is practically very difficult for $\sum_{k=1}^r \sigma_k = \sum_{k=1}^r d_k$ to hold as this requires $\sum_{i=1}^r x_{ki}^2 = \sum_{i=1}^r y_{ki}^2 = 1$ and $x_{ki} = y_{ki}$, for all $k, i \in [r]$. This means that conducting singular value decomposition to any $Z = \sum_{k=1}^r d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^\intercal$ and substituting singular values and vectors into P2 can typically get a better result, which indicates that $\boldsymbol{\alpha}_{1:r}$ and $\boldsymbol{\beta}_{1:r}$

Table 1: Results on different missing rates

Setting	$m = 500, n = 500, r = 30, q = 5$			
Missing-Rates	90%	80%	50%	0%
BPMF	1.6842 ± 0.1374	0.3210 ± 0.0168	0.1304 ± 0.0022	0.0933 ± 0.0000
GASR	0.1992 ± 0.0241	0.1321 ± 0.0086	0.0841 ± 0.0028	0.0724 ± 0.0036
Setting	$m = 1000, n = 1000, r = 50, q = 10$			
Missing-Rates	90%	80%	50%	0%
BPMF	0.9422 ± 0.0478	0.2396 ± 0.0033	0.1105 ± 0.0013	0.0859 ± 0.0007
GASR	0.2513 ± 0.0045	0.1688 ± 0.0041	0.1270 ± 0.0034	0.1115 ± 0.0057

will nearly always get orthonormalized automatically under the unit sphere constraints.

$\square$

Remark 1. *The optimal solutions for P2 are not necessarily unique. As a result, the optimal $\boldsymbol{\alpha}_{1:r}$ and $\boldsymbol{\beta}_{1:r}$ for P2 are not always orthonormalized, though orthonormal vectors provide a solution. However, what matters is not the orthonormality of $\boldsymbol{\alpha}_{1:r}$ and $\boldsymbol{\beta}_{1:r}$, but the equivalence of optimal solution $Z = \sum_{k=1}^{r} d_k \boldsymbol{\alpha}_k \boldsymbol{\beta}_k^{\intercal}$. Theorem 1 asserts that P1 and P2 produce the same set of optimal matrix completion results. As a result, the MAP problem constructed according to P2 is anticipated to function similarly as the one constructed from P1.*

Note that the condition for strict equality in $\sum_{k=1}^{r} \sigma_k^* \leq \sum_{k=1}^{r} d_k^*$ is practically very hard to satisfy.

Remark 2. *This special relationship between P1 and P2 in Theorem 1 can be generalized to other forms of noise potentials besides the squared-error loss as well as the max-margin hinge loss used in MMMF, as we do not need the property of $\|\cdot\|_F$ in our proof. The theorem should still hold if we replace $\|\cdot\|_F$ with $\|\cdot\|_1$, $\|\cdot\|_\infty$, etc.*

Detailed Experimental Results for Different Missing Rates

In this experiment, we run both BPMF and GASR for 100 iterations and average all 100 samples to produce the final result. The average RMSE and corresponding deviations on 3 randomly generated datasets are reported in Table 1.